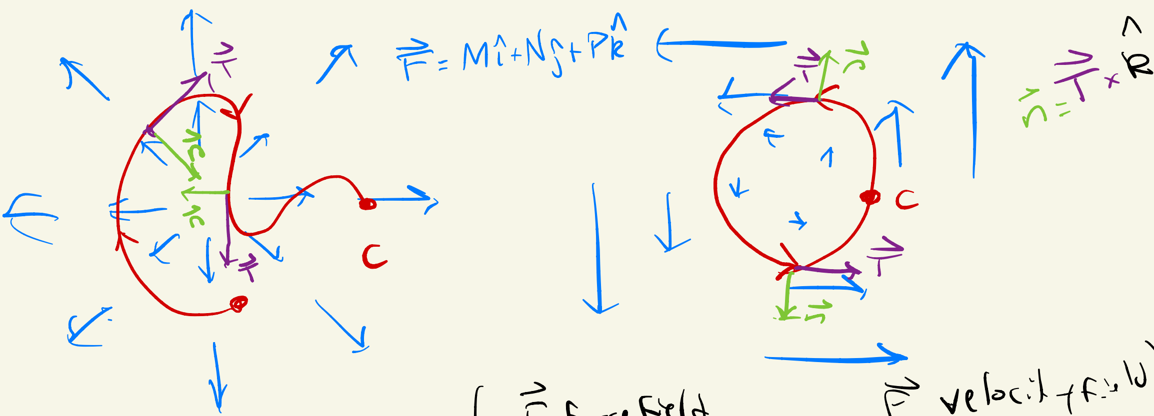
8-23

Last time ... 2 flavors of integrals of curves.

Take a vector field $\vec{F}$ (eg. force field like gravity or velocity field like water current)

and oriented curve C



· Line integrals: ($\vec{F}$ force field
$=$ work done by $\vec{F}$, $\vec{F}$ velocity field
$=$ flow of $\vec{F}$) ← if C a closed curve: circulation

$$\int_C \vec{F} \cdot \vec{T} \, ds \overset{(*)}{=} \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

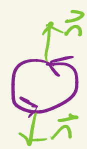
(*) need parametrization $\vec{r}(t)$ $a \le t \le b$

· Flux integrals:

$$\int_C \vec{F} \cdot \vec{n} \, ds = \oint_C M\,dy - N\,dx$$

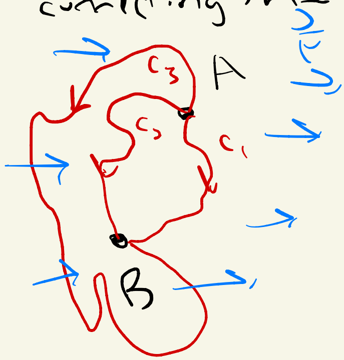
outward flux

if C oriented ccw and closed $\oint$



Special vector fields:

- Conservative vector fields $\vec{F}$ have path-independent line integrals — "obey conservation of energy" since for any two points A, B, every line integral connecting them only depends on A, B:

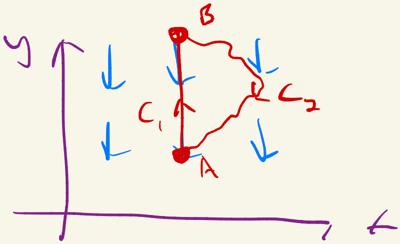


$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}$$

Means it only depends on coordinates for A and B!

Leads us to: potential functions!

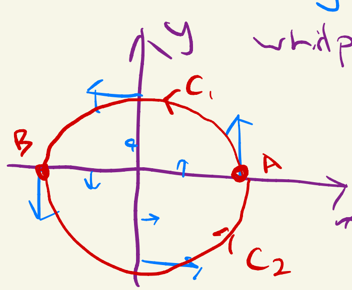
$\vec{F} = -\hat{j}$ Gravity field



$\vec{F}$ conservative $\Leftrightarrow$ $\vec{F}$ has a potential function

$\vec{F} = -y\hat{i} + x\hat{j}$

whirlpool



$\vec{F}$ not conservative $\Leftrightarrow$ $\vec{F}$ has no potential function

Def: Potential

If $\vec{F}$ is a vector field defined on an open region $D \subseteq \mathbb{R}^n$ $(n=2,3)$ and $\vec{F} = \nabla f$ for some scalar function f on D, then we call f the <u>potential function</u> for $\vec{F}$

what will happen?

Idea: 1D: $f(x) \xrightarrow{\text{"Derivative"}} f'(x)$

(nD) 2D or 3D: $\underbrace{f(x,y,z)}_{\text{scalar function}} \xrightarrow{\text{"Derivative"}} \underbrace{\nabla f}_{\text{vector field}}$

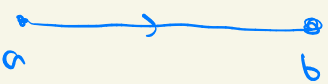
In 21A:

Fundamental theorem of calc:

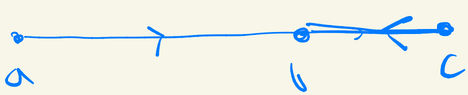
$$\int_a^b \underbrace{f'(x)}_{\text{gradient}}\,dx = \underbrace{f(b)-f(a)}_{\text{potential function}}$$

is like a 1D line integral:

$$\vec{F} = f'(x)\,\hat{i}$$



in 1D, this integral is path independent:



$$\int_a^c f'(x)\,dx + \int_c^b f'(x)\,dx = \int_a^b f'(x)\,dx$$

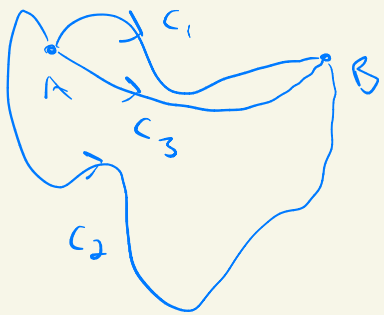
$$= f(b) - f(a)$$

different path didn't affect the integral

⇒ path independent (after checking all paths)

⇒ $\vec{F}$ is conservative and has a potential function $f(x)$.

- Now, in 2 and 3D, want to "fundamental thm of calc" to move from ∇f to f.
- Story gets richer: many more paths in 2 and 3D between A and B!



want something like

$$\int_A^B \underbrace{\qquad\qquad}_{\text{"Derivative"}} \cdot d\vec{r} = f(B) - f(A)$$

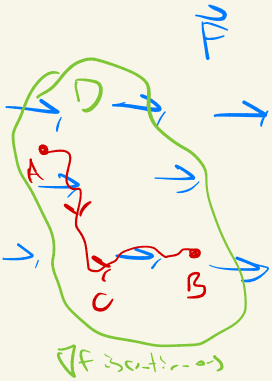
will need path independence for this to work

Fundamental Thm of Line Integrals (Gradient Thm)

Let C be a smooth oriented curve joining A to $B \in \mathbb{R}^2$ or $\mathbb{R}^3$ and parameterized by $\vec{r}(t)$.

Let f be a differentiable function with a continuous gradient vector field ∇f on a domain $D \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ containing C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



[Fund Thm of Calc is special 1D case:

$\nabla f = f'(x)$

$d\vec{r} = dx$

$\Rightarrow \int_C f'(x)\,dx = f(B) - f(A)$

Consequence: $\vec{F} = \nabla f$ is conservative on D.

2nd consequence: (contrapositive of consequence)

if a vector field $\vec{F}$ is not conservative, it cannot be a gradient field (there is no potential))

Ex:

Sps the force field $\vec{F}$ is the gradient of the function

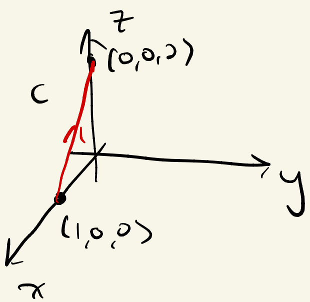
$$f(x,y,z) = \frac{-1}{x^2+y^2+z^2}$$ (Gravity potential)

Find the work done by $\vec{F}$ in moving an object along a smooth curve C joining $(1,0,0)$ to $(0,0,2)$ avoiding $(0,0,0)$.

Soln:

First: notice that since $\vec{F} = \nabla f$, by consequence of gradient thm $\vec{F}$ is conservative $\Rightarrow$ this work integral will be path independent

Option 1:



pick an easy path C, then by gradient thm, it's path independent so

$$\int_C \nabla f \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r}$$

Option 2: use gradient thm

$$\int_C \nabla f \cdot d\vec{r} = \underbrace{f(0,0,2)}_{f(B)} - \underbrace{f(1,0,0)}_{f(A)}$$

$$= -\frac{1}{4} - (-1)$$

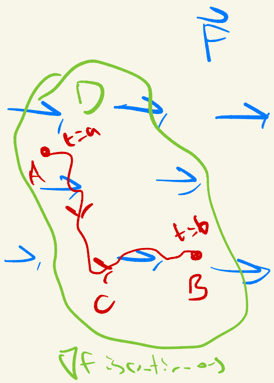
$$= \frac{3}{4}$$

Makes sense because we are farther from the origin!! after field works on us there's more potential energy

Pf of Gradient thm:

Let A, B in D. C a smooth oriented curve joining them

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$
$$a \le t \le b$$



Note: chain rule for $f(x,y,z)$

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \frac{\partial f(\vec{r}(t))}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f(\vec{r}(t))}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f(\vec{r}(t))}{\partial z}\cdot\frac{\partial z}{\partial t}$$
$$\overset{(*)}{=} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$\int_C \nabla f \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r}$$
$$= \int_{t=a}^{t=b} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$
$$\overset{(*)}{=} \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt$$

Fundamental thm of calc

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$
$$= f(B) - f(A) \qquad \blacksquare$$

Notice:

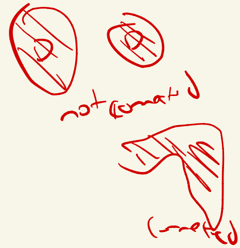
By considering earlier,
- we can't integrate (find a potential fn) nonconservative vector fields
- Thm tells us gradient fields can be integrated.
  what about other conservative vector fields?

Thm (Conservative fields are Gradient fields)

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ a vector field whose components are continuous throughout an open connected region D in space.

Then

$$\vec{F} \text{ is conservative} \iff \vec{F} = \nabla f \text{ for some } f$$



Note:

- If you suspect $\vec{F}$ is conservative, one way to prove it is to find a potential fn!
- Continuous partials are important here! Discontinuities can make problems. Look at problem 5 ps 7.

Ex:

Let

$$\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

Calculate the work done along the line connecting $A=(-1,3,9)$ to $B=(1,6,-4)$

soln:

Looks like it may come from a potential. Let's guess and check (will have a better method later):

$$f(x,y,z) = xyz$$

$$\Rightarrow \vec{F} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \nabla f$$

Thm $\Rightarrow \vec{F}$ is conservative

So, by gradient thm,

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r}$$

gradient thm $\rightarrow$ $$= f(B) - f(A)$$

$$= xyz\Big|_{(1,6,-4)} - xyz\Big|_{(-1,3,9)}$$

$$= (1)(6)(-4) - (-1)(3)(9)$$

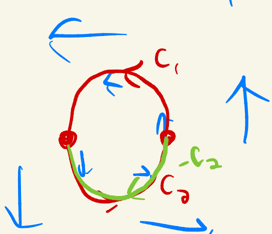
$$= -24 + 27$$

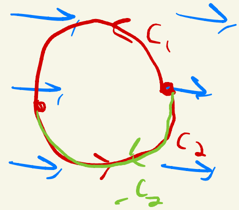
$$= 3$$

Regarding next thm:

Two examples:

$\vec{F} = -y\hat{i} + x\hat{j}$ $\qquad\qquad$ $\vec{F} = \hat{i}$





whirl pool (not conservative) $\qquad\qquad$ Lazy river (conservative)

$$\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{-C_2} \vec{F}\cdot d\vec{r} \qquad\qquad \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-C_2} \vec{F}\cdot d\vec{r}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = 0$$

$$\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} \neq 0$$

Thm: Loop property of conservative fields

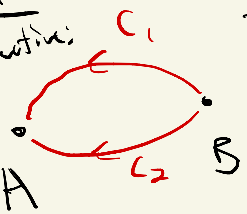
$$\oint_C \vec{F} \cdot d\vec{r} = 0 \quad \text{over every closed loop } C \text{ in domain } D$$

$$\Updownarrow$$

The field $\vec{F}$ is conservative on D

Pf:

Conservative:

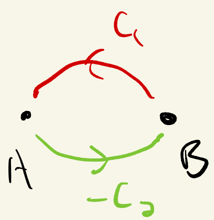


For all C_1, C_2

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

(path independence)

$$\Updownarrow$$



For all C_1, C_2,

$$\int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} \quad \text{(loop property)}$$

$\blacksquare$

To summarize:

$\vec{F} = \nabla f$ on D $\Longleftrightarrow$ $\vec{F}$ conservative on D (path independence)

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

over any loop in D

So:

1. How do we know whether a given $\vec{F}$ is conservative?
   (component test/curl test for conservative fields)

2. If $\vec{F}$ is conservative, how do we find a potential function f s.t. $\vec{F} = \nabla f$?

Thm: Component test / curl test for conservative fields

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field on an open simply connected domain whose component functions have continuous first partial derivatives. Then

$$\vec{F} \text{ is conservative} \iff \left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial N}{\partial z} \\ \frac{\partial M}{\partial z} &= \frac{\partial P}{\partial x} \\ \frac{\partial N}{\partial x} &= \frac{\partial M}{\partial y} \end{aligned}\right\} \text{all hold } (*)$$

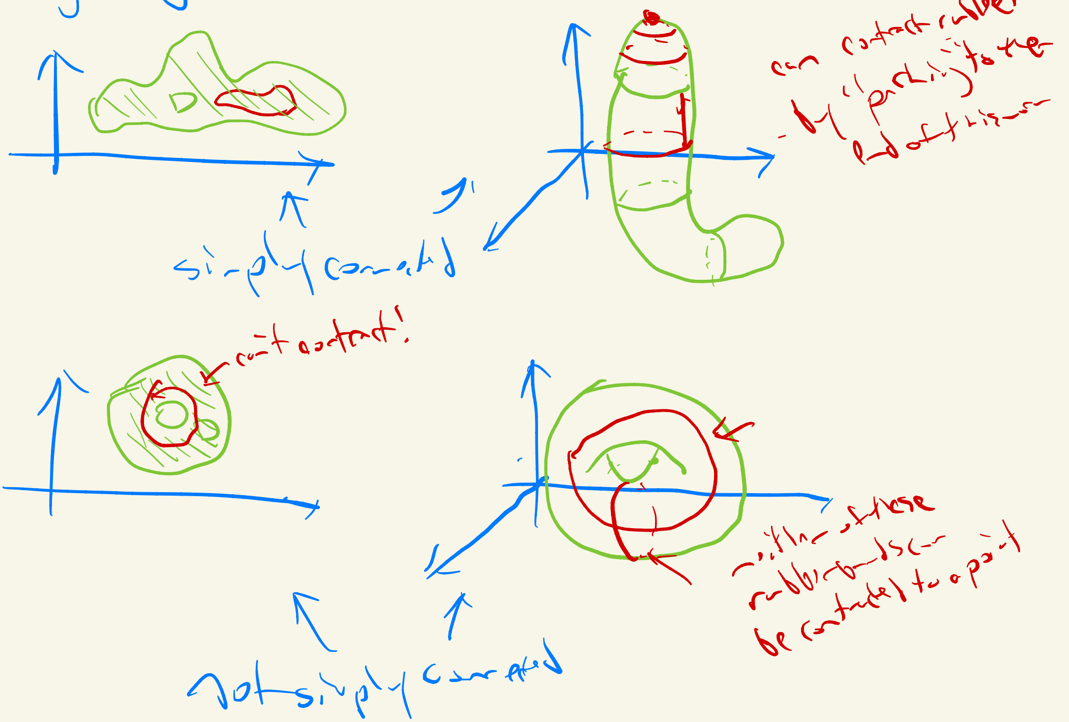
$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\Rightarrow \operatorname{curl}\vec{F} = 0 \iff (*) \text{ all hold}$$

Simply Connected

D is simply connected means every loop in D can be contracted to a single point without ripping or gluing: (imagine rubber band living in D)



Potential problem from not simply connected:

problem 5 PS7

singularity at (0,0) causes the problem!

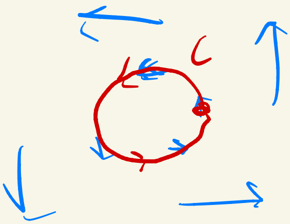
$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$ · on unit circle, since $x^2+y^2=1$, $\vec{F}\big|_{\text{unit circle}} = -y\hat{i}+x\hat{j}$ (whirlpool)

Curl $\vec{F}$ detects rotation:

Recall whirlpool

$$\vec{F} = -y\hat{i} + x\hat{j} + 0\hat{k}$$

We know it isn't conservative



$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0$$

Check:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right)\hat{i}$$

$$- \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right)\hat{k}$$

$$= 2\hat{k}$$

Nonzero curl = "rotation present"

= not conservative!

Finding a potential:

$\vec{F}$ conservative $\iff$ there is a potential function f with $\nabla f = \vec{F}$

1. Check whether $\vec{F}$ is conservative with component/curl test

2. If it is conservative, then we can solve
$$\nabla f = \vec{F}$$
$$\iff \frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P$$

So integrate a bunch!

Ex:
Show $\vec{F} = (e^x \cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy + z)\hat{k}$ is conservative over its natural domain and find a potential function for $\vec{F}$.

Sol:
$$\operatorname{curl} \vec{F} = (x - x)\hat{i} - (y - y)\hat{j} + (-e^x \sin y + z - (-e^x \sin y + z))\hat{k}$$

All partials are continuous, and $\operatorname{curl} \vec{F} = 0$
$\Rightarrow$ conservative.

Next time (during E.C. tuesday):

will find $\vec{F}$'s potential.